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Viscous rotational stagnation-point flow

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The investigation by Hayes (1964*a*) of the behaviour of a constant-density inviscid rotational flow in the neighbourhood of a stagnation point on a plane wall has been extended to include the effects of viscosity. The principal effect is the manner in which the singularity in vorticity discovered by Hayes is removed. A solution of only the boundary-layer equatiosn indicates the vorticity decays algebraically from the wall. Application of the method of matched asymptotic expansions, however, shows that the difference between boundary layer and outer vorticity, when carried out to second order in the outer flow, does not contribute to an algebraic decay. These results suggest that an infinite number of higher-order outer terms are generated which match the algebraic terms thereby yielding the conventional exponential decay. Numerical results are presented which also support this conclusion. The main contribution of the wall shear stress in the immediate neighbourhood of the stagnation point is shown to come from the external lateral vorticity.

1. Introduction

In a recent paper Hayes (1964*a*) investigated the behaviour of a constant-density inviscid rotational flow in the neighbourhood of a stagnation point on a plane wall. His solutions show that the distribution of lateral vorticity in the flow approaching the wall is amplified by stretching of vortex lines and becomes singular at the wall. The purpose of this paper is to enquire into the nature of the corrections to the lateral vorticity components when the fluid is viscous.

Hayes adopted the model that the lateral velocities were linear functions of the lateral variables; the normal velocity being a function of only the normal variable. He shows that the equations governing the motion of the flow split into two parts, one of which he calls primary and the other, secondary. The primary set of equations is to be interpreted as the basic flow approaching the wall; the secondary equations represent the convection of the lateral vorticity towards the wall by the primary flow. In this sense, the secondary flow is dependent upon the primary flow; the primary flow, however, is independent of the secondary flow. The flow is characterized by the basic parameter α_0, $0 \leqslant \alpha_0 \leqslant 1$, which is a measure of its departure from axial symmetry. For $\alpha_0 = 0$, the flow is axisymmetric or almost axisymmetric (i.e. the primary flow is axisymmetric, but the combined primary and secondary flow may be asymmetric); for $0 < \alpha_0 < 1$, it is asymmetric; and for $\alpha_0 = 1$, it is planar or almost-planar. In another paper

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Hayes (1964*b*) shows that under suitable restrictions, solutions for $\alpha_0 > 1$ also exist. This case represents flow directed inward along one lateral co-ordinate towards the stagnation point. Along the other co-ordinate the flow is directed away from the stagnation point. Only one lateral component of vorticity, however, is permitted with the vorticity decreasing as it is convected towards the wall.

2. Basic equations

We use the flow field and notation of Hayes (1964*a*). The space of interest is the part of the physical space with Cartesian co-ordinates (x, y, z) for which $z \geqslant 0$ with the wall the plane $z = 0$. The velocity field is

$$\mathbf{q} = \tfrac{1}{2}U'a[F + x(H' - \alpha),\ G + y(H' + \alpha), -2H], \tag{2.1}$$

where F, G, H, and α are functions of z alone and the quantities U' and a are the reference velocity gradient characterizing the normal flow and the reference length, respectively. The vorticity corresponding to (2.1) is

$$\nabla \times \mathbf{q} = \tfrac{1}{2}U'[-G' - y(H'' + \alpha'), F' + x(H'' - \alpha'), 0]. \tag{2.2}$$

By inserting (2.1) into the momentum equation, and setting the curl of the pressure gradient identically equal to zero, Hayes derives the following set of equations:

$$R^{-1}\alpha'' + H\alpha' - H'\alpha = A, \tag{2.3a}$$

$$R^{-1}H''' + HH'' - \tfrac{1}{2}(H'^2 + \alpha^2) = D, \tag{2.3b}$$

$$R^{-1}F'' + HF' - \tfrac{1}{2}(H' - \alpha)F = M, \tag{2.3c}$$

$$R^{-1}G'' + HG' - \tfrac{1}{2}(H' + \alpha)G = N, \tag{2.3d}$$

where A, D, M, and N are constants and $R = U'a^2/\nu$ is the Reynolds number. We call, following Hayes, the solutions to (2.3*a*, *b*) the primary flow and the solutions to (2.3*c*, *d*) the secondary flow.

3. The effects of viscosity

In considering the viscous corrections, we follow the procedure described by Van Dyke (1964) and solve the problem in terms of matched asymptotic expansions.

3.1. *The outer expansion*

It is well known that stagnation-point flow solutions are local in the sense that they are valid only in the immediate neighbourhood of the stagnation point. Thus the question of how the external flow specification arises cannot be asked. It follows from this that in considering the viscous corrections, the external flow should be specified in as simple a manner as possible.

With this viewpoint we seek solutions subject to the constraint that the primary external flow be irrotational. This means that the higher-order terms of the inviscid solutions to H, α, F, and G found by Hayes (1964*a*) can be ignored. In addition, other higher-order terms, due to local body curvature, discovered by

Boger & Ludford (1967) [see also Boger 1966] will also be ignored. By avoiding the complications arising from the existence of these higher-order terms, we simplify considerably the solution to the external flow.

The form of the outer expansion has been suggested by Hayes who showed the first-order inner flow to be boundary layer in character. Thus, the correction to the basic external flow arises from the displacement thickness of the boundary layer which is known to be proportional to $R^{-\frac{1}{2}}$. We anticipate this by introducing a displaced outer co-ordinate

$$z' = z - \beta R^{-\frac{1}{2}}, \tag{3.1}$$

where β is the boundary-layer displacement thickness of the primary flow. Keeping the quantities H', α, F, and G invariant with respect to this transformation, the form of the equations describing the outer flow is identical to the equations (2.3*a–d*).

We expand the outer variables in powers of $\epsilon^2 = R^{-1}$ where now the first term represents the combined basic inviscid flow and the correction due to the boundary-layer displacement thickness,

$$H(z'; R) = H_1(z') + \epsilon^2 H_2(z') + \ldots, \tag{3.2a}$$

$$\alpha(z'; R) = A_1(z') + \epsilon^2 A_2(z') + \ldots, \tag{3.2b}$$

$$F(z'; R) = F_1(z') + \epsilon^2 F_2(z') + \ldots, \tag{3.2c}$$

$$G(z'; R) = G_1(z') + \epsilon^2 G_2(z') + \ldots. \tag{3.2d}$$

The equations describing the outer flow are obtained by substituting (3.2*a–d*) into (2.3*a–d*) and collecting the coefficients of the various powers of ϵ. To second order, the equations are:

First order

$$H_1 A_1' - H_1' A_1 = A, \tag{3.3a}$$

$$H_1 H_1'' - \tfrac{1}{2}[(H_1')^2 + A_1^2] = D, \tag{3.3b}$$

$$H_1 F_1' - \tfrac{1}{2}(H_1' - A_1) F_1 = M, \tag{3.3c}$$

$$H_1 G_1' - \tfrac{1}{2}(H_1' + A_1) G_1 = N. \tag{3.3d}$$

Second order

$$H_1 A_2' - H_1' A_2 = -H_2 A_1' + H_2' A_1 - A_1'', \tag{3.4a}$$

$$H_1 H_2'' - H_1' H_2' + H_1'' H_2 = A_1 A_2 - H_1''', \tag{3.4b}$$

$$H_1 F_2' - \tfrac{1}{2}(H_1' - A_1) F_2 = H_2 F_1' - \tfrac{1}{2}(H_2' - A_2) F_1 - F_1'', \tag{3.4c}$$

$$H_1 G_2' - \tfrac{1}{2}(H_1' + A_1) G_2 = -H_2 G_1' - \tfrac{1}{2}(H_1' + A_2) G_1 - G_1''. \tag{3.4d}$$

3.2. *Outer flow solutions*

First order (basic inviscid flow). Equations (3.3*a–d*) must satisfy at the displaced wall $z' = 0$ the boundary conditions

$$H_1'(0) = 1, \quad A_1(0) = \alpha_0, \quad F_1(0) = G_1(0) = 0. \tag{3.5a–d}$$

The only solution to (3.3*a–d*) that both satisfies these boundary conditions and restricts the primary flow to be irrotational is given by

$$H_1(z') = z', \quad A_1(z') = \alpha_0, \quad F_1(z') = f_0 z'^{\frac{1}{2}(1-\alpha_0)}, \quad G_1(z') = g_0 z'^{\frac{1}{2}(1+\alpha_0)}, \tag{3.6a–d}$$

where the constants f_0 and g_0 are arbitrary. They are determined by prescribing values of the secondary flow F and G at some height z'_0 above the displaced wall. The constants A, D, M, and N, evaluated by inserting the solution given by (3.6) into (3.3), are

$$A = \alpha_0, \quad D = -\tfrac{1}{2}(1+\alpha_0^2), \quad M = N = 0. \tag{3.7a–d}$$

Equation (3.6) represents a special case of Hayes' (1964*a*) more general solution. It is valid for $0 \leqslant \alpha_0 < 1$. In the following analysis, we restrict the solutions to values of α_0 given by $0 \leqslant \alpha_0 < 1$.

It is worth noting, however, that by making suitable restrictions, Hayes (1964*b*) shows that a solution for the case $\alpha_0 > 1$ also exists. Here the flow along the x co-ordinate is directed inward towards the stagnation point; along the y co-ordinate, the flow is directed outward away from the stagnation point. For this case, however, the constant $f_0 = 0$, otherwise the solution to (3.6*c*) would not satisfy the boundary condition given by (3.5*c*). The primary solution is equivalent to the case $c < 0$ discussed by Davey (1961) where the flow streamline pattern has a saddle-point behaviour at the displaced wall $z' = 0$.

Second order. The solution to (3.4*a–d*) that satisfies the constraint that the primary flow be irrotational is

$$A_2(z') = 0, \tag{3.8a}$$

$$H_2(z') = 0, \tag{3.8b}$$

$$F_2(z') = -\tfrac{1}{8}(1-\alpha_0^2) f_0 z'^{-\frac{1}{2}(3+\alpha_0)} + F_0 z'^{\frac{1}{2}(1-\alpha_0)}, \tag{3.8c}$$

$$G_2(z') = -\tfrac{1}{8}(1-\alpha_0^2) g_0 z'^{-\frac{1}{2}(3-\alpha_0)} + G_0 z'^{\frac{1}{2}(1+\alpha_0)}. \tag{3.8d}$$

The constants F_0 and G_0 are related to the second-order outer vorticity distribution. A description of such a vorticity distribution is beyond the scope of this paper, hence the constants are unspecified.

3.3. *The inner expansion*

To investigate the inner solution, we introduce the usual stretching of the (unshifted) normal co-ordinate z and the normal velocity H, while keeping the lateral co-ordinates and the lateral velocities invariant, by the transformation

$$\eta = R^{\frac{1}{2}} z \quad \text{and} \quad H(\eta) = R^{\frac{1}{2}} H(z). \tag{3.9a, b}$$

With the equations of motion transformed, we expand the velocities in terms of powers of $\epsilon = R^{-\frac{1}{2}}$ where the leading term in the expansion represents the boundary-layer solution. The form of the inner expansion is suggested by rewriting the outer solutions in inner variables (i.e. $z' = \epsilon(\eta - \beta)$) and expanding for large R.

Thus, the inner expansion is written

$$H(\eta; R) = h_1(\eta) + \ldots, \tag{3.10a}$$

$$\alpha(\eta; R) = \alpha_1(\eta) + \ldots, \tag{3.10b}$$

$$F(\eta; R) = \epsilon^{\frac{1}{2}(1-\alpha_0)} f_1(\eta) + \epsilon^{\frac{1}{2}(5-\alpha_0)} f_2(\eta) + \ldots, \tag{3.10c}$$

$$G(\eta; R) = \epsilon^{\frac{1}{2}(1+\alpha_0)} g_1(\eta) + \epsilon^{\frac{1}{2}(5+\alpha_0)} g_2(\eta) + \ldots. \tag{3.10d}$$

What is new here is that the inner expansion is characterized by fractional powers of ϵ instead of the usual integer powers. The fractional powers are representative of the three dimensionality of the flow approaching the wall. Recall that for $\alpha_0 = 0$, the flow is symmetric, or almost symmetric and that for $\alpha_0 = 1$, the flow is planar, or almost planar.

The boundary conditions at the wall are the usual no-slip conditions

$$h(0) = h'(0) = \alpha(0) = f(0) = g(0) = 0. \qquad (3.11\,a\text{–}e)$$

The remaining boundary conditions far from the wall are to be determined by matching to the outer flow.

Substituting these expansions into (2.3) and collecting coefficients of the various powers of ϵ yields the following set of equations to second order:

First-order (boundary-layer) equations

$$\alpha_1'' + h_1\alpha_1' - h_1'\alpha_1 + \alpha_0 = 0, \qquad (3.12a)$$

$$h_1''' + h_1 h_1'' - \tfrac{1}{2}[(h_1')^2 + \alpha_1^2] + \tfrac{1}{2}(1+\alpha_0^2) = 0, \qquad (3.12b)$$

$$f_1'' + h_1 f_1' - \tfrac{1}{2}(h_1' - \alpha_1) f_1 = 0, \qquad (3.12c)$$

$$g_1'' + h_1 g_1' - \tfrac{1}{2}(h_1' + \alpha_1) g_1 = 0. \qquad (3.12d)$$

Second-order equations

$$f_2'' + h_1 f_2' - \tfrac{1}{2}(h_1' - \alpha_1) f_2 = 0, \qquad (3.13a)$$

$$g_1'' + h_1 g_2' - \tfrac{1}{2}(h_1' + \alpha_1) g_2 = 0. \qquad (3.13b)$$

3.4. *Matching of inner and outer expansions*

We will now investigate the asymptotic behaviour of the inner and outer expansions to ensure that the assumed form of the expansions given by (3.2) and (3.10) are compatible.

First-order (boundary-layer) equations. The asymptotic behaviour of the primary flow is well known. The inner solutions decay exponentially to their outer behaviour with the form

$$h_1(\eta) \sim \eta - \beta + \exp \quad \text{and} \quad \alpha_1(\eta) \sim \alpha_0 + \exp. \qquad (3.14a, b)$$

Here β is the boundary-layer displacement thickness. The asymptotic behaviour of the secondary flow, however, is not so well known. Replacing h_1 and α_1 for large η by (3.14a, b), the equations describing the motion of the secondary velocities become

$$f_1'' + (\eta-\beta) f_1' - \tfrac{1}{2}(1-\alpha_0) f_1 = 0 \quad \text{and} \quad g_1'' + (\eta-\beta) g_1' - \tfrac{1}{2}(1+\alpha_0) g_1 = 0. \quad (3.15a, b)$$

To investigate the asymptotic behaviour of (3.15a, b), we transform them to confluent hypergeometric differential equations by introducing the transformation

$$\xi = -\tfrac{1}{2}(\eta-\beta)^2. \qquad (3.16)$$

Equations (3.15a, b) are transformed to

$$\xi f_1'' + (\tfrac{1}{2} - \xi) f_1' + \tfrac{1}{4}(1-\alpha_0) f_1 = 0 \quad \text{and} \quad \xi g_1'' + (\tfrac{1}{2} - \xi) g_1' + \tfrac{1}{4}(1+\alpha_0) g_1 = 0, \quad (3.17a, b)$$

where f_1 and g_1 are invariant under the transformation. Their asymptotic behaviour is given by Slater (1964), for $0 \leqslant \alpha_0 < 1$, as

$$f_1(\eta) \sim C_1 \eta^{\frac{1}{2}(1-\alpha_0)}[1-\tfrac{1}{2}(1-\alpha_0)\beta\eta^{-1}-\tfrac{1}{8}(1-\alpha_0^2)(1+\beta^2)\eta^{-2}+O(\eta^{-3})]+\exp \quad (3.18a)$$

and

$$g_1(\eta) \sim C_2 \eta^{\frac{1}{2}(1+\alpha_0)}[1-\tfrac{1}{2}(1+\alpha_0)\beta\eta^{-1}-\tfrac{1}{8}(1-\alpha_0^2)(1+\beta^2)\eta^{-2}+O(\eta^{-3})]+\exp, \quad (3.18b)$$

where the constants C_1 and C_2 are to be determined by matching to the first-order outer flow.

Upon matching the outer behaviour of the first-order inner solution to the inner behaviour of the first-order outer solution, the constants C_1 and C_2 become

$$C_1 = f_0 \quad \text{and} \quad C_2 = g_0. \quad (3.19a, b)$$

Second-order solution. To investigate the asymptotic behaviour of the second-order secondary velocities, we replace h_1 and α_1 for η sufficiently large, by (3.14a, b) to yield the simplified second-order equations

$$f_2'' + (\eta-\beta)f_2' - \tfrac{1}{2}(1-\alpha_0)f_2 = 0, \quad (3.20a)$$

$$g_2'' + (\eta-\beta)g_2' - \tfrac{1}{2}(1+\alpha_0)g_2 = 0. \quad (3.20b)$$

The asymptotic behaviour of the secondary solutions is precisely the same as the first-order solutions given by (3.18a, b) except for a redefinition of the constants of integration. Upon matching the outer behaviour of the two-term inner solution to the inner behaviour of the two-term outer solution, the constants are found to be equal to the constants F_0 and G_0 defined by (3.8c, d).

3.5. *Asymptotic decay of the viscous flow*

An examination of the asymptotic behaviour of the secondary flow indicates that it appears to decay algebraically from the wall. We will show, however, that the inner behaviour of the higher-order outer terms match to the outer behaviour of the inner terms to yield the conventional exponential decay. To show this, we will consider in detail the asymptotic behaviour of the first-order secondary velocity $f_1(\eta; R)$. The other term $g_1(\eta; R)$ can be shown to decay exponentially in a similar manner.

The outer behaviour of $f_1(\eta; R)$ is given by (3.18a). The inner behaviour of the outer flow is determined by rewriting the outer flow in inner variables and expanding for large Reynolds number. The outer flow, determined from (3.2c), (3.6c), and (3.8c) where $\epsilon = R^{-\frac{1}{2}}$, is

$$F(z; R) = f_0 z'^{\frac{1}{2}(1-\alpha_0)} + \epsilon^2[-\tfrac{1}{8}(1-\alpha_0^2)f_0 z'^{-\frac{1}{2}(3+\alpha_0)} + F_0 z'^{\frac{1}{2}(1-\alpha_0)}]. \quad (3.21)$$

Rewriting (3.21) in inner co-ordinates (i.e. $z' = \epsilon(\eta-\beta)$) and retaining only the leading terms results in

$$F(\eta; R) = \epsilon^{\frac{1}{2}(1-\alpha_0)} f_0 \eta^{\frac{1}{2}(1-\alpha_0)}\left[1-\frac{(1-\alpha_0)\beta}{2\eta}-\frac{(1-\alpha_0^2)(1+\beta^2)}{8\eta^2}+O(\eta^{-3})\right]. \quad (3.22)$$

Comparing (3.22) with (3.18a) shows that the difference between the boundary-layer and outer secondary velocity, when carried out to second order in the outer flow, decays exponentially. Although we have not proved that all the terms in (3.18a) match asymptotically to higher-order outer terms, the analysis suggests that they do.

The results of this study are in agreement with the results reached by Conti & Van Dyke (1969) in a related study. They considered reacting flow near a blunt stagnation point where the inviscid flow has a general algebraic type of

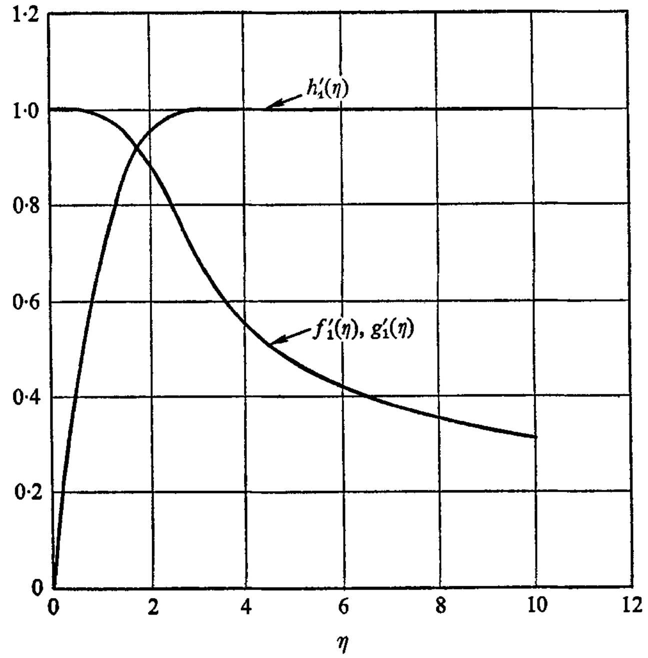

FIGURE 1. Velocity and vorticity profiles for the case $\alpha_0 = 0$.

singularity. They showed that the apparent algebraic decrease in vorticity in the second-order boundary layer precisely matches a corresponding behaviour in the second-order outer flow. Thus, they found that the difference between boundary layer and outer vorticity, when carried out to second order in the outer flow, does not contribute to an algebraic decay.

4. Numerical results

Numerical solutions to the primary and secondary boundary-layer equations were obtained using a Runge–Kutta integration procedure accurate to four decimal places. Results valid for $\alpha_0 = 0$, 0·3, and 0·5 are presented in figures 1 to 3. It is evident that the primary solutions decay exponentially to their outer values with a strong dependence upon α_0. The sensitivity to α_0 is clearly illustrated in figure 4 which shows the behaviour, with α_0, of the displacement thickness β.

The numerical results also suggest that the asymptotic behaviour of the secondary flow decays exponentially to its outer values. To show this we have plotted in figure 5 the 1-term and the 3-term outer expansion rewritten in inner

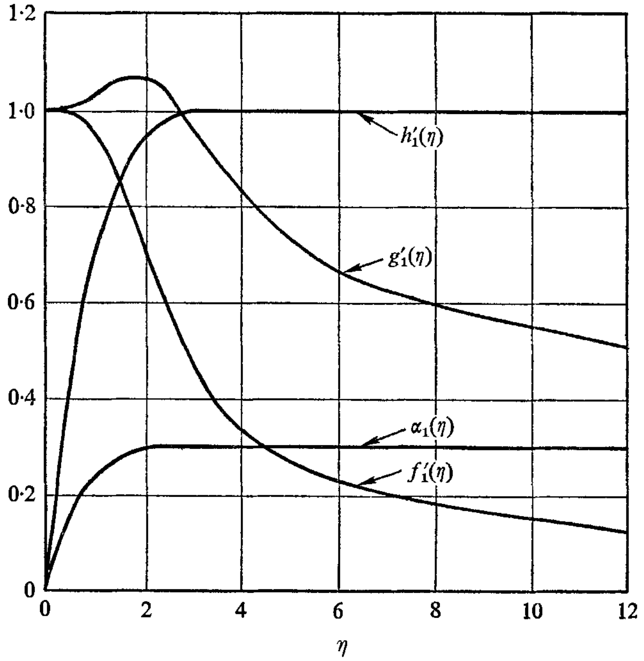

FIGURE 2. Velocity and vorticity profiles for the case $\alpha_0 = 0{\cdot}3$.

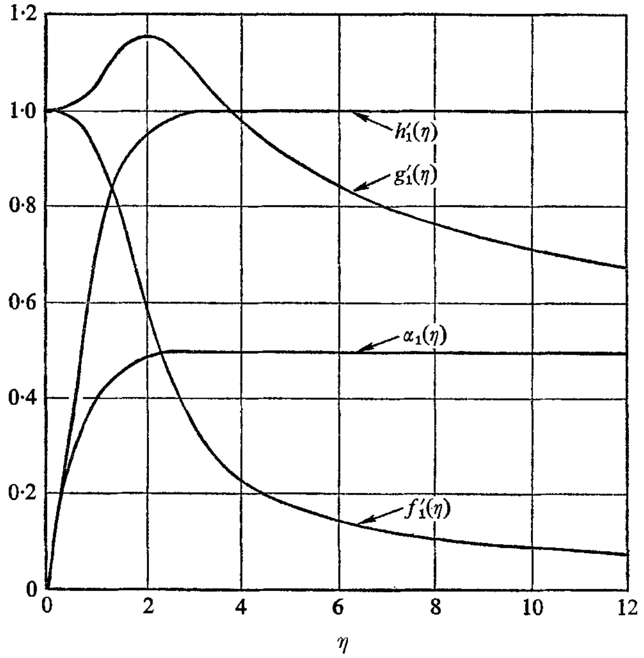

FIGURE 3. Velocity and vorticity profiles for the case $\alpha_0 = 0{\cdot}5$.

variables from (3.22) for the case $\alpha_0 = 0$. The constant $\epsilon^{\frac{1}{2}} f_0$ was selected to match the 3-term outer expansion to the inner numerical solution at $\eta = 40$. The 3-term outer expansion appears to match quite satisfactorily to the inner solution for $\eta > 2$. The 1-term outer expansion, however, appears to match to the inner solution only for extremely large values of η. Thus the 1-term outer expansion suggests an algebraic decay while the 3-term outer expansion suggests an exponential decay. These conclusions are consistent with the conclusions of § 3.5.

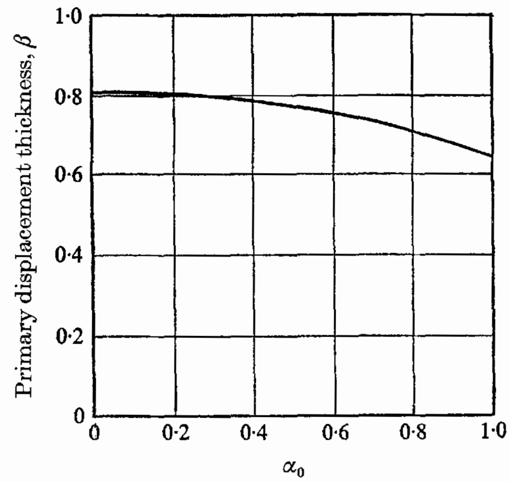

FIGURE 4. The behaviour of the primary flow displacement thickness with α_0.

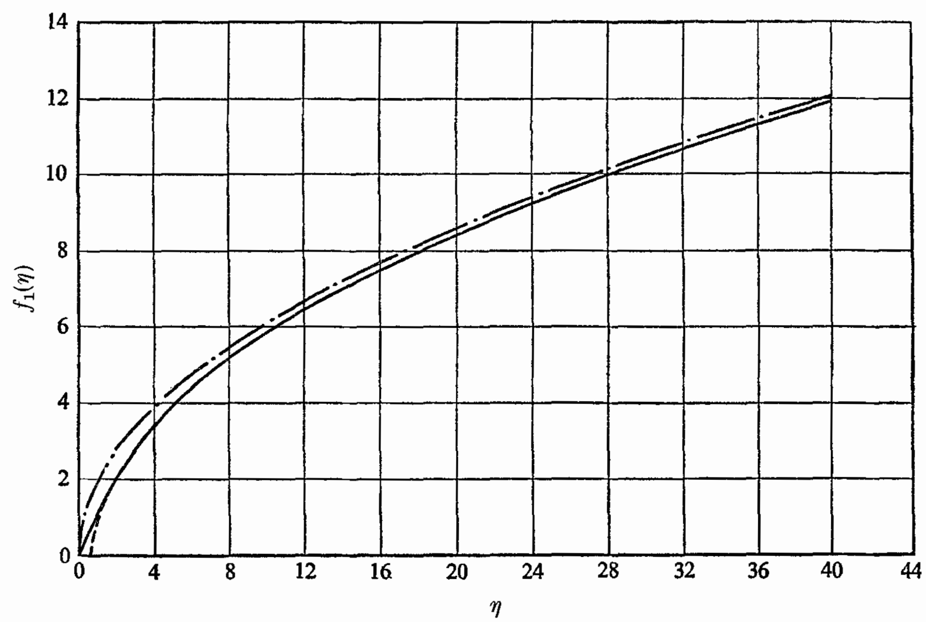

FIGURE 5. Matching between the inner and outer secondary solutions for the case $\alpha_0 = 0$. —, numerical solution; — - —, 1-term outer expansion; – – –, 3-term outer expansion. The 3-term outer expansion is indistinguishable from the numerical solution for $\eta > 2$.

It is of interest to consider the individual contributions to the wall shear stress from the primary and secondary flows. The skin friction coefficients in the x and y directions are defined as

$$c_{fx} = \frac{\tau_{xz}}{\frac{1}{2}\rho(U'a)^2}\bigg|_{z=0} = \frac{2}{R^{\frac{1}{2}}}\left\{\frac{f_1'(0)}{R^{\frac{1}{2}(1-\alpha_0)}} + x[h_1''(0) - \alpha_1'(0)]\right\}, \tag{4.1}$$

and

$$c_{fy} = \frac{\tau_{yz}}{\frac{1}{2}\rho(U'a)^2}\bigg|_{z=0} = \frac{2}{R^{\frac{1}{2}}}\left\{\frac{g_1'(0)}{R^{\frac{1}{2}(1+\alpha_0)}} + y[h_1''(0) + \alpha_1'(0)]\right\}, \tag{4.2}$$

where only the leading (boundary-layer) terms in the expansion for h, a, f, and g have been retained. It is evident from (4.1) and (4.2) that in the immediate neighbourhood of the stagnation point, the majority of the shear stress comes

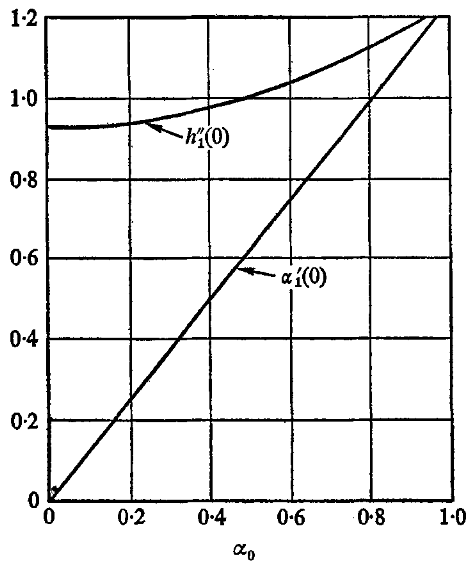

FIGURE 6. The behaviour of the primary flow wall shear stress with α_0.

from the secondary flow. Moreover, the shear stresses are strongly dependent upon the parameter α_0. The relationship between the primary shear stresses at the wall with the parameter α_0 is shown in figure 6. An equivalent unique relationship for the secondary flow is not possible because their equations are linear, hence the shear stresses are proportional to their outer values. It is for this reason that the secondary shear stresses at the wall $\eta = 0$ have been normalized to unity.

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